College and Career Readiness Standards for Mathematics

Draft for Review and Comment

July 16, 2009

Contents

Exemplars for the draft Math Standards can be found at
http://www.corestandards.net/mathexemplars.html

Introduction and Overview of the Organization

Ten *Mathematical Principles* form the backbone of these standards. Each principle is accompanied by an explanation that describes the coherent view students are expected to have of a specific area of mathematics. With this coherent view, students will be better able to learn more mathematics and use the mathematics they know. The principles pull together topics previously studied and target topics yet to be learned in post-secondary programs.

Each principle consists of a statement of a Coherent Understanding of the principle, together with Core Concepts, Core Skills, and Explanatory Problems that exemplify and delimit the range of tasks students should be able to do.

These standards, like vectors, specify direction and distance for students to be ready for college and careers:

1. Direction—The Coherent Understanding

   The Coherent Understandings attempt to communicate the mathematical coherence of the knowledge students should take into college and careers. They are intended to tell teachers, 'This is how your students should see the mathematics in this area in order to aim them towards mastering it.'

2. Distance—The Concepts, Skills and Explanatory Problems

   Collectively, these statements and sets of problems define and clarify the level of expertise students should reach if they are to be prepared for success in college and career. They are

   a. statements of concepts students must know and actions students must be able to take using the mathematics; and

   b. examples of the problems and other assignments they must be able to complete.

In addition to the *Mathematical Principles*, the standards also contain a set of *Mathematical Practices* that are key to using mathematics in the workplace, in further education and in a 21st Century democracy. Students who care about being precise, who look for hidden structure and note regularity in repeated reasoning, who make sense of complex problems and persevere in solving them, who construct viable arguments and use technology intelligently are more likely to be able to apply the knowledge they have attained in school to new situations. These mathematical practices are described and tied to examples.

Taken together, the explanations of the mathematical principles, the associated concepts and skills and the mathematical practices form the College and Career Readiness Standards for Mathematics.

Overview of the Mathematical Principles

Number. Procedural fluency in operations with real numbers and strategic competence in approximation are grounded in an understanding of place value. The rules of arithmetic govern operations and are the foundation of algebra.

Expressions. Expressions use symbols and efficient notational conventions about order of operations, fractions and exponents to express verbal descriptions of computations in a compact form.

Equations. An equation is a statement that two expressions are equal, which may result from expressing the same quantity in two different ways, or from asking when two different quantities have the same value. Solving an equation means finding the values of the variables in it that make it true.

Functions. Functions describe the dependence of one quantity on another. For example, the return on an investment is a function of the interest rate. Because nature and society are full of dependencies, functions are important tools in the construction of mathematical models.

Quantity. A quantity is an attribute of an object or phenomenon that can be measured using numbers. Specifying a quantity pairs a number with a unit of measure, such as 2.7 centimeters, 42 questions or 28 miles per gallon.

Modeling. Modeling uses mathematics to help us make sense of the real world—to understand quantitative relationships, make predictions, and propose solutions.

Shape. Shapes, their attributes, and the relations among them can be analyzed and generalized using the deductive method first developed by Euclid, generating a rich body of theorems from a few axioms.

Coordinates. Applying a coordinate system to Euclidean space connects algebra and geometry, resulting in powerful methods of analysis and problem solving.

Probability. Probability assesses the likelihood of an event. It allows for the quantification of uncertainty, describing the degree of certainty that an event will happen as a number from 0 through 1.

Statistics. We often base decisions or predictions on data. The decisions or predictions would be easy to make if the data always sent a clear signal, but the signal is usually obscured by noise. Statistical analysis aims to account for both the signal and the noise, allowing decisions to be as well informed as possible.

College and Career Readiness Standards for Mathematics

Mathematical Practices

Proficient students expect mathematics to make sense. They take an active stance in solving mathematical problems. When faced with a non-routine problem, they have the courage to plunge in and try something, and they have the procedural and conceptual tools to carry through. They are experimenters and inventors, and can adapt known strategies to new problems. They think strategically. The mathematical practices described below bind together the five strands of mathematical proficiency: procedural fluency, conceptual understanding, strategic competence, adaptive reasoning, and productive disposition.[a]

Students who engage in these practices discover ideas and gain insights that spur them to pursue mathematics beyond the classroom walls.[b] They learn that effort counts in mathematical achievement.[c] These are practices that expert mathematical thinkers encourage in apprentices. Encouraging these practices should be as much a goal of the mathematics curriculum as is teaching specific content topics and procedures.[d]

1. They care about being precise.

Mathematically proficient students organize their own ideas in a way that can be communicated precisely to others, and they analyze and evaluate others' mathematical thinking and strategies based on the assumptions made. They clarify definitions. They state the meaning of the symbols they choose, are careful about specifying units of measure and labeling axes, and express their answers with an appropriate degree of precision. They would never say "let v be speed and let t be elapsed time" but rather "let v be the speed in meters per second and let t be the elapsed time in seconds." They recognize that when someone says the population of the United States in June 2008 was 304,059,724, the last few digits are meaningless.

2. They construct viable arguments.

Mathematically proficient students understand and use stated assumptions, definitions and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They break things down into cases and can recognize and use counterexamples. They use logic to justify their conclusions, communicate them to others and respond to the arguments of others.

3. They make sense of complex problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for the entry points to its solution. They consider analogous problems, try special cases and work on simpler forms. They evaluate their progress and change course if necessary. They try putting algebraic expressions into different forms or try changing the viewing window on their calculator to get the information they need. They look for correspondences between equations, verbal descriptions, tables, and graphs. They draw diagrams of relationships, graph data, search for regularity and trends, and construct mathematical models. They check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?"

4. They look for structure.

Mathematically proficient students look closely to discern a pattern or stand back to get an overview or shift their perspective, and they transfer fluently between these points of view. For example, in $x^2 + 5x + 6$ they can see the 5 as $2 + 3$ and the 6 as 2×3 They recognize the significance of an existing line in a geometric figure or add an auxiliary line to make the solution of a problem clear. They also can step back and see complicated things, such as some algebraic expressions, as single objects that they can manipulate. For example, they might determine that the value of $5 - 3(x - y)^2$ is at most 5 because $(x - y)^2$ is non-negative.[d]

5. They look for and express regularity in repeated reasoning.

Mathematically proficient students pay attention to repeated calculations as they are carrying them out, and look both for general algorithms and for shortcuts. For example, by paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, they might abstract an equation of the line of the form $\frac{y-2}{x-1} = 3$. By noticing the telescoping in the expansions of $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$, they might derive the general formula for the sum of a geometric series. As they work through the solution to a problem, they maintain oversight over the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.[d]

6. They make strategic decisions about the use of technological tools.

Mathematically proficient students consider the available tools when solving a mathematical problem, whether pencil and paper, graphing calculators, spreadsheets, dynamic geometry or statistical software. They are familiar enough with all of these tools to make sound decisions about when each might be helpful. They use mathematical understanding, attention to levels of precision and estimation to provide realistic levels of approximation and to detect possible errors.

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a) The term *proficiency* is used here as it was defined in the 2001 National Research Council report *Adding it up: Helping children learn mathematics*. The term was used in the same way by the National Mathematics Advisory Panel (2008).

b) Singapore standards

c) National Mathematics Advisory Panel (2008)

d) Cuoco, A. , Goldenberg, E. P., and Mark, J. (1996). *Journal of Mathematical Behavior*, 15 (4), 375-402; *Focus in High School Mathematics*. Reston, VA: NCTM, in press.

Number

Core Concepts · Students understand that:

A Standard algorithms are based on place value and the rules of arithmetic.

B Fractions represent numbers. Equivalent fractions have the same value.

C All real numbers can be located on the number line.

A Coherent Understanding of Number. Procedural fluency in operations with real numbers and strategic competence in approximation are grounded in an understanding of place value. The rules of arithmetic govern operations and are the foundation of algebra.

The place value system bundles units into 10s, then 10s into 100s, and so on, providing a method for naming large numbers. Subdividing in a similar way extends this to the decimal system for naming all real numbers and locating them on a number line. This system is the basis for efficient algorithms. Numbers represented as fractions, such as rational numbers, can also be located on the number line by seeing them as numbers expressed in different units (for example, 3/5 is three fifths).

Operations with fractions depend on applying the rules of arithmetic:

- Numbers can be added in any order with any grouping and multiplied in any order with any grouping.
- Multiplication by 1 and addition of 0 leave numbers unchanged.
- All numbers have additive inverses, and all numbers except zero have multiplicative inverses.
- Multiplication distributes over addition.

Mental computation strategies are opportunistic uses of these rules, which, for example, allow one to compute the product $5 \times 177 \times 2$ at a glance, obtaining 1770 instantly rather than methodically working from left to right.

Sometimes an estimate is more appropriate than an exact value. For example, it might be more useful to give the length of a board approximately as 1 ft $4\frac{3}{4}$ in, rather than exactly as $\sqrt{2}$ ft; an estimate of how long a light bulb lasts helps in determining the number of light bulbs to buy. In addition, estimation and approximation are useful in checking calculations.

Connections to Expression, Equations and Functions. The rules of arithmetic govern the manipulations of expressions and functions and, along with the properties of equality, provide a foundation for solving equations.

Core Skills · Students can and do:

1 Use standard algorithms with procedural fluency.*

2 Use mental strategies and technology with strategic competence.**

3 Compare numbers and make sense of their magnitude.

Include positive and negative numbers expressed as fractions, decimals, powers and roots. Limit to square and cube roots. Include very large and very small numbers.

4 Solve multi-step problems involving fractions and percentages.

Include situations such as simple interest, tax, markups/markdowns, gratuities and commissions, fees, percent increase or decrease, percent error, expressing rent as a percentage of take-home pay, and so on. Students should also be able to solve problems of the three basic forms: 25 percent of 12 is what? 3 is what percent of 12? and 3 is 25 percent of what? and understand how these three problems are related.

5 Use estimation to solve problems and detect errors.

6 Give answers to an appropriate level of precision.

* The term procedural fluency as used in this document has the same meaning as in the National Research Council report *Adding it up: Helping children learn mathematics*. Specifically, "*Procedural fluency* refers to knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently" (p. 121).

** The term strategic competence as used in this document has the same meaning as in the National Research Council report *Adding it up: Helping children learn mathematics*. Specifically, "*Strategic competence* refers to the ability to formulate mathematical problems, represent them, and solve them" (p. 124).

Expressions

Core Concepts · Students understand that:

A Expressions represent computations with symbols standing for numbers.

B Complex expressions are made up of simpler expressions.

C Rewriting expressions serves a purpose in solving problems.

A Coherent Understanding of Expressions. Expressions use symbols and efficient notational conventions about order of operations, fractions and exponents to express verbal descriptions of computations in a compact form.

For example, $p + 0.05p$ expresses the addition of a 5% tax to a price p. Reading an expression with comprehension involves analysis of its underlying structure, which may suggest a different but equivalent way of writing it that exhibits some different aspect of its meaning. For example, rewriting $p + 0.05p$ as $1.05p$ shows that adding a tax is the same as multiplying by a constant factor.

Heuristic mnemonic devices are not a substitute for procedural fluency, which depends on understanding the basis of manipulations in the rules of arithmetic and the conventions of algebraic notation. For example, factoring, expanding, collecting like terms, the rules for interpreting minus signs next to parenthetical sums, and adding fractions with a common denominator are all instances of the distributive law; the interpretation we give to negative and rational exponents is based on the extension of the exponent laws for positive integers to negative and rational exponents. When simple expressions within more complex expressions are treated as single quantities, or chunks, the underlying structure of the larger expression may be more evident.

Connections to Equations and Functions. Setting expressions equal to each other leads to equations. Expressions can define functions, with equivalent expressions defining the same function.

Core Skills · Students can and do:

1 See structure in expressions and manipulate simple expressions with procedural fluency.

See Explanatory Problems.

2 Write an expression to represent a quantity in a problem.

3 Interpret an expression and its parts in terms of the quantity it represents.

See Explanatory Problems.

Equations

Core Concepts · Students understand that:

A An equation is a statement that two expressions are equal.

B Solving is a process of algebraic manipulation guided by logical reasoning.

C Completing the square leads to a formula for solving quadratic equations.

D Equations not solvable in one number system might be solvable in a larger system.

A Coherent Understanding of Equations. An equation is a statement that two expressions are equal, which may result from expressing the same quantity in two different ways, or from asking when two different quantities have the same value. Solving an equation means finding the values of the variables in it that make it true.

The solutions of an equation in one variable form a set of numbers; the solutions of an equation in two variables form a set of ordered pairs, which can be graphed in the plane. Equations can be combined into systems to be solved simultaneously.

An equation can be solved by successively transforming it into one or more simpler equations. The process is governed by deductions based on the properties of equality. For example, one can add the same constant to both sides without changing the solutions, but squaring both sides might lead to extraneous solutions. Some equations have no solutions in a given number system, stimulating the formation of expanded number systems (integers, rational numbers, real numbers and complex numbers). Strategic competence in solving includes looking ahead for productive manipulations and anticipating the nature and number of solutions.

A formula expressing a general relationship among several variables is a type of equation, and the same solution techniques used to solve equations can be used to rearrange formulas. For example, the formula for the area of a trapezoid, $A = \left(\frac{b_1+b_2}{2}\right)h$, can be solved for h using the same deductive steps.

Like equations, inequalities can involve one or more variables and can be solved in much the same way. Many, but not all, of the properties of equality extend to the solution of inequalities.

Connections to Functions, Coordinates, and Modeling. Equations in two variables can define functions, and questions about when two functions have the same value lead to equations. Graphing the functions allows for the approximate solution of equations. Equations of lines are addressed under Coordinates, and converting verbal descriptions to equations is addressed under Modeling.

Core Skills · Students can and do:

1 Understand a word problem and restate it as an equation.

Extend to inequalities and systems.

2 Solve equations in one variable using manipulations guided by the rules of arithmetic and the properties of equality.

Solve linear equations with procedural fluency. For quadratic equations, include solution by inspection, by factoring, or by using the quadratic formula. *See Explanatory Problems.*

3 Rearrange formulas to isolate a quantity of interest.

Exclude cases that require extraction of roots or inverse functions.*

4 Solve systems of equations.

Focus on pairs of simultaneous linear equations in two variables. Include algebraic techniques, graphical techniques and solving by inspection.

5 Solve linear inequalities in one variable and graph the solution set on a number line.

6 Graph the solution set of a linear inequality in two variables on the coordinate plane.

* Exclusions of this sort are modeled after Singapore's standards, which contains similar exclusions and limitations to help define the desired level of complexity.

Functions

Core Concepts · Students understand that:

A A function describes the dependence of one quantity on another.

B The graph of a function *f* is a set of ordered pairs $(x, f(x))$ in the coordinate plane.

C Common functions occur in parametric families where each member describes a similar type of dependence.

A Coherent Understanding of Functions. Functions describe the dependence of one quantity on another. For example, the return on an investment is a function of the interest rate. Because nature and society are full of dependencies, functions are important tools in the construction of mathematical models.

Functions in school mathematics are often presented by an algebraic rule. For example, the time in hours it takes for a plane to fly 1000 miles is a function of the plane's speed in miles per hour; the rule $T(s) = 1000/s$ expresses this dependence algebraically and is said to define a function, whose name is *T*. The graph of a function is a useful way of visualizing the dependency it models, and manipulating the expression for a function can throw light on its properties. Sometimes functions are defined by a recursive process which can be modeled effectively using a spreadsheet or other technology.

Two important families of functions are characterized by laws of growth: linear functions grow at a constant rate, and exponential functions grow at a constant percent rate. Linear functions with an initial value of zero describe proportional relationships.

Connections to Expressions, Equations, Modeling and Coordinates. Functions may be defined by expressions. The graph of a function *f* is the same as the solution set of the equation $y = f(x)$. Questions about when two functions have the same value lead to equations, whose solutions can be visualized from the intersection of the graphs. Since functions express relationships between quantities, they are frequently used in modeling.

Core Skills · Students can and do:

1 Recognize proportional relationships and solve problems involving rates and ratios.

2 Describe the qualitative behavior of common types of functions using expressions, graphs and tables.

Use graphs and tables to identify: intercepts; intervals where the function is increasing, decreasing, positive or negative; relative maximums and minimums; symmetries; and periodicity. Explore the effects of parameter changes (including shifts and stretches) on the graphs of these functions using technology. Include linear, power, quadratic, polynomial, simple rational, exponential, logarithmic, trigonometric, absolute value and step functions. *See Explanatory Problems.*

3 Analyze functions using symbolic manipulation.

Include slope-intercept and point-slope form of linear functions; factored form to find horizontal intercepts; vertex form of quadratic functions to find maximums and minimums; and manipulations as described under Expressions. *See Explanatory Problems.*

4 Use the families of linear and exponential functions to solve problems.

For linear functions $f(x) = mx + b$, understand *b* as the intercept or initial value and m as the slope or rate of change. For exponential functions $f(x) = a{\cdot}b^x$, understand *a* as the intercept or initial value and *b* as the growth factor. *See Explanatory Problems.*

5 Find and interpret rates of change.

Compute the rate of change of a linear function and make qualitative observations about the rates of change of nonlinear functions.

Quantity

Core Concepts · Students understand that:

A The value of a quantity is not specified unless the units are named or understood from the context.

B Quantities can be added and subtracted only when they are of the same general kind (lengths, areas, speeds, etc.).

C Quantities can be multiplied or divided to create new types of quantities, called derived quantities.

A Coherent Understanding of Quantity. A quantity is an attribute of an object or phenomenon that can be measured using numbers. Specifying a quantity pairs a number with a unit of measure, such as 2.7 centimeters, 42 questions or 28 miles per gallon.

For example, the length of a football field and the speed of light are both quantities. If we choose units of miles per second, then the speed of light has the value 186,000 miles per second. But the speed of light need not be expressed in second per hour; it may be expressed in meters per second or any other unit of speed. A speed of 186,000 miles per second is the same as a speed of meters per second. "Bare" numerical values such as 186,000 and do not describe quantities unless they are paired with units.

Speed (distance divided by time), rectangular area (length multiplied by length), density (mass divided by volume), and population density (number of people divided by area) are examples of derived quantities, obtained by multiplying or dividing quantities.

It can make sense to add two quantities, such as when a child 51 inches tall grows 3 inches to become 54 inches tall. To be added or subtracted, quantities must be expressed in the same units, but even then it does not always make sense to add them. If a wooded park with 300 trees per acre is next to a field with 30 trees per acre, they do not have 330 trees per acre. Converting quantities to have the same units is like converting fractions to have a common denominator before adding or subtracting.

Doing algebra with units in a calculation reveals the units of the answer, and can help reveal a mistake if, for example, the answer comes out to be a distance when it should be a speed.

Connections to Number, Expressions, Equations, Functions and Modeling. Operations described under Number and Expressions govern the operations one performs on quantities, including the units involved. Quantity is an integral part of any application of mathematics, and has connections to solving problems using equations, functions and modeling.

Core Skills · Students can and do:

1 Use units consistently in describing real-life measures, including in data displays and graphs.

2 Know when and how to convert units in computations.

Include the addition and subtraction of quantities of the same general kind expressed in different units; averaging data given in mixed units; converting units for derived quantities such as density and speed.

3 Use and interpret derived quantities and units correctly in algebraic formulas.

4 Use units as a way to understand problems and to guide the solution of multi-step problems.

Include examples such as acceleration; currency conversions; people-hours; social science measures, such as deaths per 100,000; and general rate, such as points per game. *See Explanatory Problems.*

Modeling

Core Concepts · Students understand that:

A Models abstract key features from situations to help us solve problems.

B Models can be useful even if their assumptions are oversimplified.

A Coherent Understanding of Modeling. Modeling uses mathematics to help us make sense of the real world—to understand quantitative relationships, make predictions, and propose solutions.

A model can be very simple, such as a geometric shape to describe a physical object like a coin. Even so simple a model involves making choices. It is up to us whether to model the solid nature of the coin with a three-dimensional cylinder, or whether a two-dimensional disk works well enough for our purposes. For some purposes, we might even choose to adjust the right circular cylinder to model more closely the way the coin deviates from the cylinder.

In any given situation, the model we devise depends on a number of factors: How exact an answer do we want or need? What aspects of the situation do we most need to understand, control, or optimize? What resources of time and tools do we have? The range of models we can create and analyze is constrained as well by the limitations of our mathematical and technical skills. For example, modeling a physical object, a delivery route, a production schedule, or a comparison of loan amortizations each requires different sets of tools. Networks, spreadsheets and algebra are powerful tools for understanding and solving problems drawn from different types of real-world situations.

The basic modeling cycle is one of (1) apprehending the important features of a situation, (2) creating a mathematical model that describes the situation, (3) analyzing and performing the mathematics needed to draw conclusions from the model, and (4) interpreting the results of the mathematics in terms of the original situation.

Connections to Quantity, Equations, Functions, Shape and Statistics. Modeling makes use of shape, data and algebra to represent real-world quantities and situations. In this way the Modeling Principle relies on concepts of quantity, equations, functions, shape and statistics.

Core Skills · Students can and do:

1 Model numerical situations.

Include readily applying the four basic operations in combination to solve multi-step quantitative problems with dimensioned quantities; making estimates to introduce numbers into a situation and get a problem started; recognizing proportional or near-proportional relationships and analyzing them using characteristic rates and ratios.

2 Model physical objects with geometric shapes.

Include common objects that can reasonably be idealized as two- and three-dimensional geometric shapes. Identify the ways in which the actual shape varies from the idealized geometric model.

3 Model situations with equations, inequalities and functions.

Include situations well described by a linear inequality in two variables or a system of linear inequalities that define a region in the plane; situations well described by linear, quadratic or exponential equations or functions; and situations that can be well described by inverse variation.

4 Model situations with common functions.

Include identifying a family that models a problem and identify a particular function of that family adjusting parameters. Understand the recursive nature of situations modeled by linear and exponential functions.

5 Model data with statistics.

Include replacing a distribution of values with a measure of its central tendency; modeling a bivariate relationship using a trend line or a linear regression line.

6 Compare models for a situation.

Include recognizing that there can be many models that relate to a situation, that they can capture different aspects of the situation, that they can be simpler or more complex, and that they can have a better or worse fit to the situation and the questions being asked.

7 Interpret the results of applying the model in the context of the situation.

Include realizing that models do not always fit exactly and so there can be error; identifying simple sources of error and being careful not to over-interpret models.

Shape

Core Concepts · Students understand that:

A Shapes, their attributes, and their measurements can be analyzed deductively.

B Right triangles and the Pythagorean theorem are focal points in geometry with practical and theoretical importance.

C Congruent shapes can be superimposed through rigid transformations.

D Proportionality governs the relationship between measurements of similar shapes.

A Coherent Understanding of Shape. Shapes, their attributes, and the relations among them can be analyzed and generalized using the deductive method first developed by Euclid, generating a rich body of theorems from a few axioms.

The analysis of an object rests on recognition of the points, lines and surfaces that define its shape: a circle is a set of points in a plane equidistant from a fixed point; a cube is a figure composed of six identical square regions in a particular three-dimensional arrangement. Precise definitions support an understanding of the ideal, allowing application to the real world where geometric modeling, measurement, and spatial reasoning offer ways to interpret and describe physical environments.

We can also analyze shapes, and the relations of congruence and symmetry, through transformations such as translations, reflections, and rotations. For example, the line of reflective symmetry in an isosceles triangle assures that its base angles are equal.

The study of similar right triangles supports the definition of sine, cosine and tangent for acute angles, and the Pythagorean theorem is a key link between shape, measurement, and coordinates. Knowledge about triangles and measurement can be applied in practical problems, such as estimating the amount of wood needed to frame a sloping roof.

Connections to Coordinates and Functions. The Pythagorean theorem provides an important bridge between shape and distance in the coordinate plane. Parameter changes in families of functions can be interpreted as transformations applied to their graphs.

Core Skills · Students can and do:

1 Use geometric properties to solve multi-step problems involving shapes.

Include: measures of angles of a triangle sum to 180°; measures of vertical, alternate interior and corresponding angles are equal; measures of supplemental angles sum to 180°; two lines parallel to a third are parallel to each other; points on a perpendicular bisector of a segment are equidistant from the segment's endpoints; and the radius of a circle is perpendicular to the tangent at the point of intersection of the circle and radius. *See Explanatory Problems.*

2 Prove theorems, test conjectures and identify logical errors.

Include theorems about angles, parallel and perpendicular lines, similarity and congruence of triangles.

3 Solve problems involving measurements.

Include measurement (length, angle measure, area, surface area, and volume) of a variety of figures and shapes in two- and three-dimensions. Compute measurements using formulas and by decomposing complex shapes into simpler ones. *See Explanatory Problems.*

4 Construct shapes from a specification of their properties using a variety of tools.

Include classical construction techniques and construction techniques supported by modern technologies.

5 Solve problems about similar triangles and scale drawings.

Include computing actual lengths, areas and volumes from a scale drawing and reproducing a scale drawing at a different scale.

6 Apply properties of right triangles and right triangle trigonometry to solve problems.

Include applying sine, cosine and tangent to determine lengths and angle measures of a right triangle, the Pythagorean theorem and properties of special right triangles. Use right triangles and their properties to solve real-world problems. Limit angle measures to degrees. *See Explanatory Problems.*

7 Create and interpret two-dimensional representations of three-dimensional objects.

Include schematics, perspective drawings and multiple views.

Coordinates

Core Concepts · Students understand that:

A Locations in space can be described using numbers called coordinates.

B Coordinates serve as tools for blending algebra with geometry and allow methods from one domain to be used to solve problems in the other.

C The set of solutions to an equation in two variables is a line or curve in the coordinate plane and the solutions to systems of equations in two variables correspond to intersections of lines or curves.

D Equations in different families graph as different sorts of curves—such as straight lines, parabolas, circles.

A Coherent Understanding of Coordinates. Applying a coordinate system to Euclidean space connects algebra and geometry, resulting in powerful methods of analysis and problem solving.

Just as the number line associates numbers with locations in one dimension, a pair of perpendicular axes associates pairs of numbers with locations in two dimensions. This correspondence between numerical coordinates and geometric points allows methods from algebra to be applied to geometry and vice versa. The solution set of an equation becomes a geometric curve, making visualization a tool for doing and understanding algebra. Geometric shapes can be cast as equations, making algebraic manipulation into a tool for geometric proof and understanding.

Coordinate geometry is a rich field for exploration. How does a geometric transformation such as a translation or reflection affect the coordinates of points? What features does the graph have for a rational function whose denominator can be zero? How is the geometric definition of a circle reflected in its equation?

Coordinates can also be applied to scale maps and provide a language for talking about direction and bearing. Adding a third perpendicular axis associates three numbers with locations in three dimensions and extends the use of algebraic techniques to problems involving the three-dimensional world we live in.

Connections to Shape, Quantity, Equations and Functions. Coordinates can be used to reason about shapes. In applications, coordinates often have dimensions and units (such as lengths and bushels). A one-variable equation of the form $f(x) = g(x)$ may be solved in the coordinate plane by finding intersections of the curves $y = f(x)$ and $y = g(x)$.

Core Skills · Students can and do:

1 Translate fluently between lines in the coordinate plane and their equations.

Include predicting visual features of lines by inspection of their equations, determining the equation of the line through two given points, and determining the equation of the line with a given slope passing through a given point.

2 Identify the correspondence between parameters in common families of equations and the shape of their graphs.

Include common families of equations—the graphs of $Ax + By = C$, $y = mx + b$ and $x = a$ are straight lines; the graphs of $y = a(x - h)^2 + k$ and $y = Ax^2 + Bx + C$ are parabolas; and the graph of $(x - h)^2 + (y - k)^2 = r^2$ is a circle.

3 Use coordinates to solve geometric problems.

Include proving simple theorems algebraically, using coordinates to compute perimeters and areas for triangles and rectangles, finding midpoints of line segments, finding distances between pairs of points and determining the parallelism or perpendicularity of lines. *See Explanatory Problems.*

Probability

Core Concepts · Students understand that:

A Probability expresses a rational degree of certainty with a number from 0 to 1 where probability of 1 means that an event is certain.

B When there are *n* equally likely outcomes the probability of any one of them occurring is $\frac{1}{n}$ and the probability of any combination of outcomes can be computed using the laws of probability.

C Probability is an important consideration in rational decision-making.

A Coherent Understanding of Probability. Probability assesses the likelihood of an event. It allows for the quantification of uncertainty, describing the degree of certainty that an event will happen as a number from 0 through 1.

In some situations, such as flipping a coin, rolling a number cube or drawing a card, where no bias exists for or against any particular outcome, it is reasonable to assume that the possible outcomes are all equally likely. From this assumption the laws of probability give the probability for each possible number of heads, sixes or aces after a given number of trials. Generally speaking, if you know the probabilities of some simple events you can use the laws of probability to deduce probabilities of combinations of them.

An important method in such calculations is systematically counting all the possibilities in a situation. Systematic counting often involves arranging the objects to be counted in such a way that the problem of counting reduces to a smaller problem of the same kind.

In some situations it is not known whether an event has been influenced by outside factors. If we question whether a number cube is fair, we can compare the results we get by rolling it to the frequencies predicted by the mathematical model. It is this application of probability that underpins drawing valid conclusions from sampling or experimental data. For example, if the experimental population given a drug is categorized 20 different ways, a manufacturer's claim of significant results in one of the categories is not compelling.

Connections to Statistics and Expressions. The importance of randomized experimental design provides a connection with Statistics. Probability also has a more advanced connection with the Expression principle through Pascal's triangle and binomial expansions.

Core Skills · Students can and do:

1 Use methods of systematic counting to compute probabilities.

2 Take probability into account when making decisions and solving problems.

3 Compute theoretical probabilities and compare them to empirical results.

Include one- and two-stage investigations involving simple events and their complements, compound events involving dependent and independent simple events. Include using data from simulations carried out with technology to estimate probabilities.

4 Identify and explain common misconceptions regarding probability.

Include misconceptions about long-run versus. short-run behavior (the law of large numbers) and the "high exposure fallacy" (e.g., more media coverage suggests increased probability that an event will occur, which fails to account for the fact that media covers mostly unusual events).

5 Compute probabilities from a two-way table comparing two events.

Include reading conditional probabilities from two-way tables; do not emphasize fluency with the related formulas.

Statistics

Core Concepts · Students understand that:

A Statistics quantifies the uncertainty in claims based on data.

B Random sampling and assignment open the way for statistical methods.

C Visual displays and summary statistics condense the information in large data sets.

D A statistically significant result is one that is unlikely to be due to chance.

A Coherent Understanding of Statistics. We often base decisions or predictions on data. The decisions or predictions would be easy to make if the data always sent a clear signal, but the signal is usually obscured by noise. Statistical analysis aims to account for both the signal and the noise, allowing decisions to be as well informed as possible.

We gather, display, summarize, examine and interpret data to discover patterns. Data distributions can be described by a summary statistic measuring center, such as mean or median, and a summary statistic measuring spread, such as interquartile range or standard deviation. We can compare different distributions numerically using these statistics or visually using plots. Data are not just numbers, they are numbers that mean something in a context, and the meaning of a pattern in the data depends on the context. Which statistics to compare, and what the results of a comparison may mean, depend on the question to be investigated and the real-life actions to be taken.

We can use scatter plots or two-way tables to examine relationships between variables. Sometimes, if the scatter plot is approximately linear, we model the relationship with a trend line and summarize the strength and direction of the relationship with a correlation coefficient.

We use statistics to draw inferences about questions such as the effectiveness of a medical treatment or an investment strategy. There are two important uses of randomization in inference. First, collecting data from a random sample of a population of interest clears the way for inference about the whole population. Second, randomly assigning individuals to different treatments allows comparison of the their effectiveness. Randomness is the foundation for determining the statistical significance of a claim. A statistically significant difference is one that is unlikely to be due to chance; effects that are statistically significant may, nevertheless, be small and unimportant.

Sometimes we model a statistical relationship and use that model to show various possible outcomes. Technology makes it possible to simulate many possible outcomes in a short amount of time, allowing us to see what kind of variability to expect.

Connections to Probability, Expressions, and Numbers. Inferences rely on probability. Valid conclusions about a population depend on designed statistical studies using random sampling or assignment.

Core Skills · Students can and do:

1 Identify and formulate questions that can be addressed with data; collect and organize the data to respond to the question.

2 Use appropriate displays and summary statistics for data

Include univariate, bivariate, categorical and quantitative data. Include the thoughtful selection of measures of center and spread to summarize data.

3 Estimate population statistics using samples.

Focus on the mean of the sample, and exclude standard deviation.

4 Interpret data displays and summaries critically; draw conclusions and develop recommendations.

Include paying attention to the context of the data, interpolating or extrapolating judiciously and examining the effects of extreme values of the data on summary statistics of center and spread. Include data sets that follow a normal distribution.

5 Evaluate reports based on data.

Include looking for bias or flaws in way the data were gathered or presented, as well as unwarranted conclusions, such as claims that confuse correlation with causation.

Explanatory Problems

[Note: The Explanatory Problems are incomplete in this draft. Explanatory Problems will eventually appear alongside their corresponding standards when the standards move to a two-page format.]

The purpose of the Explanatory Problems is to explain certain Core Skills and exemplify the kinds of problems students should be able to do. This feature of the College and Career Readiness Standards has been modeled on the standards of Singapore, Japan, and other high-performing countries—as well as the standards of states like Massachusetts whose standards include such problems.

Explanatory Problems have been provided for those Core Skills in which difficult judgments must be made about the desired level of mathematical complexity. For Number and Modeling, no Explanatory Problems were judged necessary to further clarify the Core Skills.

Please note that the explanatory problems are specific cases and do not fully cover the content scope of their corresponding Core Skills. Also please note that these problems are not intended to be classroom activities. They are best thought of as parts of the standards statements to which they correspond.

Number

No Explanatory Problems intended

Expressions

1 See structure in expressions and manipulate simple expressions with procedural fluency.

Explanatory Problems (a)

Perform manipulations such as the following with procedural fluency:

Expressions in (a) were taken from Japan COS, 2008

$(a+b)^2 = a^2 + 2ab + b^2$
$(a-b)^2 = a^2 - 2ab + b^2$
$(a+b)(a-b) = a^2 - b^2$
$(x+a)(x+b) = x^2 + (a+b)x + ab$

Explanatory Problems (b)

Problems in (b) were taken from Hong Kong Secondary 3 Territory-Wide Assessment 2007

Simplify $\frac{(x^4)^2}{x^2}$

Simplify $\frac{12x}{y} - \frac{3xy}{y^2}$

Explanatory Problem (c)

Problem (c) was taken from Singapore O Level January 2007 Exam

Expand fully $x\{1 - x(x+3)\}$

Additional Explanatory Problems to come

Expressions, continued

2 Write an expression to represent a quantity in a problem.

Explanatory Problems to come

Equations

2 Solve equations in one variable using manipulations guided by the rules of arithmetic and the properties of equality. Solve linear equations with procedural fluency. For quadratic equations, include solution by inspection, by factoring, or by using the quadratic formula.

Explanatory Problems to come

Functions

2 Describe the qualitative behavior of common types of functions using expressions, graphs and tables. Use graphs and tables to identify: intercepts; intervals where the function is increasing, decreasing, positive or negative; relative maximums and minimums; symmetries; and periodicity. Explore the effects of parameter changes (including shifts and stretches) on the graphs of these functions using technology. Include linear, power, quadratic, polynomial, simple rational, exponential, logarithmic, trigonometric, absolute value and step functions.

Explanatory Problems to come

3 Analyze functions using symbolic manipulation. Include slope-intercept and point-slope form of linear functions; factored form to find horizontal intercepts; vertex form of quadratic functions to find maximums and minimums; and manipulations as described under Expressions.

Explanatory Problems to come

4 Use the families of linear and exponential functions to solve problems. For linear functions $f(x) = mx + b$, understand b as the intercept or initial value and m as the slope or rate of change. For exponential functions $f(x) = a \cdot b^x$, understand a as the intercept or initial value and b as the growth factor.

Explanatory Problems to come

Quantity

4 Use units as a way to understand problems and to guide the solution of multi-step problems.
Include examples such as acceleration; currency conversions; people-hours; social science measures, such as deaths per 100,000; and general rate, such as points per game.

Explanatory Problems to come

Modeling

No Explanatory Problems intended

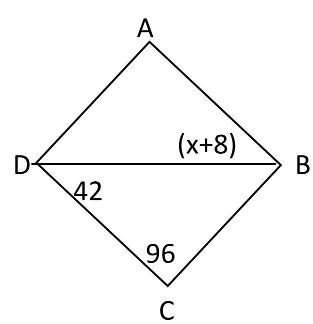
Shape

1 Use geometric properties to solve multi-step problems involving shapes. Include: measures of angles of a triangle sum to 180°; measures of vertical, alternate interior and corresponding angles are equal; measures of supplemental angles sum to 180°; two lines parallel to a third are parallel to each other; points on a perpendicular bisector of a segment are equidistant from the segment's endpoints; and the radius of a circle is perpendicular to the tangent at the point of intersection of the circle and radius.

Explanatory Problem

This problem was taken from Hong Kong Secondary 3 Territory-Wide Assessment 2007

ABCD is a rhombus. Find x. Angle measurements are in degrees.



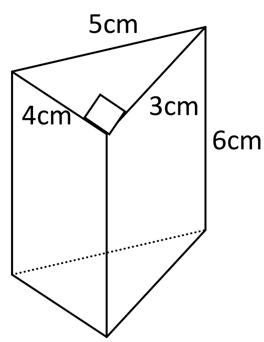
Shape, continued

3 Solve problems involving measurements. Include measurement (length, angle measure, area, surface area, and volume) of a variety of figures and shapes in two- and three-dimensions. Compute measurements using formulas and by decomposing complex shapes into simpler ones.

Explanatory Problem

This problem was taken from Hong Kong Secondary 3 Territory-Wide Assessment 2007

The figure shows a solid prism. Its base is a right-angled triangle. Find its surface area.



6 Apply properties of right triangles and right triangle trigonometry to solve problems. Include applying sine, cosine and tangent to determine lengths and angle measures of a right triangle, the Pythagorean theorem and properties of special right triangles. Use right triangles and their properties to solve real-world problems. Limit angle measures to degrees.

Explanatory Problem to come

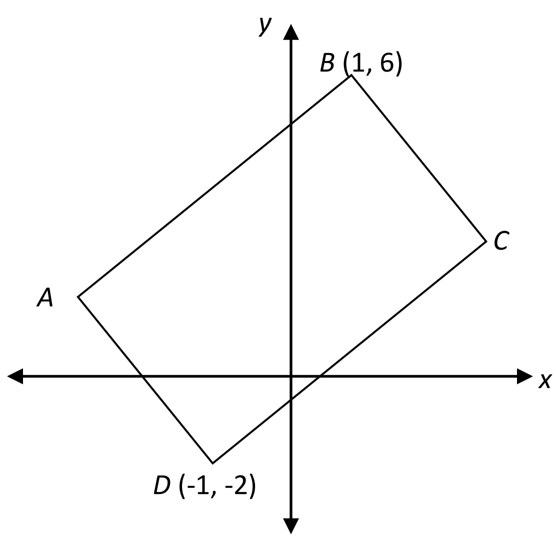
Coordinates

3 Use geometric properties to solve multi-step problems involving shapes. Include: measures of angles of a triangle sum to 180°; measures of vertical, alternate interior and corresponding angles are equal; measures of supplemental angles sum to 180°; two lines parallel to a third are parallel to each other; points on a perpendicular bisector of a segment are equidistant from the segment's endpoints; and the radius of a circle is perpendicular to the tangent at the point of intersection of the circle and radius.

Explanatory Problem

This problem was taken from Hong Kong Secondary 3 Territory-Wide Assessment 2007

The figure below shows a rectangle ABCD. Find the length of the diagonal BD of the rectangle.



Probability

1 Use methods of systematic counting to compute probabilities.

Explanatory Problems to come

2 Take probability into account when making decisions and solving problems.

Explanatory Problems to come

Statistics

2 Use appropriate displays and summary statistics for data. Include univariate, bivariate, categorical and quantitative data. Include the thoughtful selection of measures of center and spread to summarize data.

Explanatory Problems to come

5 Evaluate reports based on data. Include looking for bias or flaws in way the data were gathered or presented, as well as unwarranted conclusions, such as claims that confuse correlation with causation.

Explanatory Problems to come

How Evidence Informed Decisions in Drafting the Standards

The Common Core Standards initiative builds on a generation of standards efforts led by states and national organizations. On behalf of the states, we have taken a step toward the next generation of standards that are aligned to college- and career-ready expectations and are internationally benchmarked. These standards are grounded in evidence from many sources that shows that the next generation of standards in mathematics must be focused on deeper, more thorough understanding of more fundamental mathematical ideas and higher mastery of these fewer, more useful skills.

The evidence that supports this new direction comes from a variety of sources. International comparisons show that high performing countries focus on fewer topics and that the U.S. curriculum is "a mile wide and an inch deep." Surveys of college faculty show the need to shift away from high school courses that merely survey advanced topics, toward courses that concentrate on developing an understanding and mastery of ideas and skills that are at the core of advanced mathematics. Reviews of data on student performance show the large majority of U.S. students are not mastering the mile wide list of topics that teachers cover.

The evidence tells us that in high performing countries like Singapore, the gap between what is taught and what is learned is relatively smaller than in Malaysia or the U.S. states. Malaysia's standards are higher than Singapore's, but their performance is much lower. One could interpret the narrower gap in Singapore as evidence that they actually use their standards to manage instruction; that is, Singapore's standards were set within the reach of hard work for their system and their population. Singapore's Ministry of Education flags its webpage with the motto, "Teach Less, Learn More." We accepted the challenge of writing standards that could work that way for U.S. teachers and students: By providing focus and coherence, we could enable more learning to take place at all levels.

However, a set of standards cannot be simplistically "derived" from any body of evidence. It is more accurate to say that we used evidence to inform our decisions. A few examples will illustrate how this was done.

For example, systems of linear equations were included by all states, yet students perform surprisingly poorly on this topic when assessed by ACT. We determined that systems of linear equations have high coherence value, mathematically; that this topic is included by all high performing nations; and that it has moderately high value to college faculty. Result: We included it in our standards.

A different and more complex pattern of evidence appeared with families of functions. Again, we found that students performed poorly on problems related to many advanced functions (trigonometric, logarithmic, quadratic, exponential, and so on). Again, we found that states included

them, even though college faculty rated them lower in value. High performing countries included this material, but with different degrees of demand. We decided that we had to carve a careful line through these topics so that limited teaching resources could focus where it was most important. We decided that students should develop deep understanding and mastery of linear and simple exponential functions. They should also have familiarity (so to speak) with other families of functions, and apply their algebraic, modeling and problem solving skills to them—but not develop in-depth mastery and understanding. Thus we defined two distinct levels of attention and identified which families of functions got which level of attention.

Why were exponential functions selected in this case, instead of (say) quadratic functions? What tipped the balance was the high coherence value of exponential functions in supporting modeling and their wide utility in work and life. Quadratic functions were also judged to have received enough attention under Equations.

These examples indicate the kind of reasoning, informed by evidence, that it takes to design standards aligned to the demands of college and career readiness in a global economy. We considered inclusion in international standards, requirements of college and the workplace, surveys of college faculty and the business community, and other sources of evidence. As we navigated these sometimes conflicting signals, we always remained aware of the finiteness of instructional resources and the need for deep mathematical coherence in the standards.

In the pages that follow, the work group has identified a number of sources that played a role in the deliberations described above and more generally throughout the process to inform our decisions.

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